

The Scattering amplitude for Newly found exactly solvable Potential

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The scattering amplitude for the recently discovered exactly solvable shape invariant potential, which is isospectral to the generalized Pöschl-Taylor potential, is calculated explicitly by considering the asymptotic behavior of the X_1 Jacobi exceptional polynomials associated with this system.

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In recent years the ideas of Supersymmetric quantum mechanics (SQM) and shape invariant potentials (SIP) have greatly enriched our understanding of the exactly solvable potentials [1]. The search for the exactly solvable potentials has been boosted greatly due to the recent discovery of exceptional orthogonal polynomials (EOP) (also known as X_n Laguerre and X_n Jacobi polynomials) [2, 3]. Unlike the usual orthogonal polynomials, these EOPs start with degree $n \geq 1$ and still form a complete orthonormal set with respect to a positive definite innerproduct defined over a compact interval. This remarkable work lead Quesne [4] to the discovery of two new SIP whose solution is in terms of X_1 Laguerre and X_1 Jacobi polynomials. Subsequently, a third SIP was discovered whose solution is also in terms of X_1 Jacobi polynomials [5]. Subsequently, Odake and Sasaki constructed infinite sets of new SIP corresponding to all these three cases whose eigenfunctions are in terms of X_n Laguerre and X_n Jacobi polynomials [6].

It is worth pointing out that unlike the usual SIP, the newly discovered SIP are explicitly \hbar dependent. Further, all of them are isospectral to the well known SIP. Besides, while two out of the three newly discovered SIP have pure bound state spectrum, the third SIP which is isospectral to the generalized Pöschl-Teller (GPT) potential, has both discrete and continuous spectrum. To the best of our knowledge, while the bound state energy eigenvalues and eigenfunctions have already been obtained in all the three cases, the scattering amplitude has still not been obtained in the case of SIP which are isospectral to GPT. The purpose of this note is to fill up this gap partially. In particular, in the present work we obtain the scattering amplitude for the newly discovered SIP whose solution is in terms of X_1 Jacobi polynomial.

The superpotential corresponding to GPT (which is on the half line $0 \leq r \leq \infty$) is given by

$$W_{GPT} = A \coth r - B \operatorname{cosech} r \quad B > A + 1 > 1. \quad (1)$$

The potential $V_{GPT}(r) = W_{GPT}^2(r) - W'_{GPT}(r)$ which follows from the above super potential is given by ($\hbar = 2m = 1$)

$$V_{GPT}(r) = A^2 + [B^2 + A(A + 1)] \operatorname{cosech} r - B(2A + 1) \operatorname{cosech} r \coth r. \quad (2)$$

whose bound state energy eigenvalues and eigenfunctions are well known. Remarkably, if

we consider

$$W = W_{GPT} + \frac{2B \sinh r}{2B \cosh r - 2A - 1} - \frac{2B \sinh r}{2B \cosh r - 2A + 1}, \quad (3)$$

then we find that even though the potential $W^2(r) - W'(r)$ is very different from V_{GPT} and given by

$$V(r) = V_{GPT} + \frac{2(2A + 1)}{2B \cosh r - 2A - 1} - \frac{2[4B^2 - (2A + 1)^2]}{(2B \cosh r - 2A - 1)^2}, \quad (4)$$

the bound state spectrum is still the same and given by

$$E_\nu = A^2 - (A - \nu)^2, \quad \nu = 0, 1, \dots, \nu_{max}, \quad (5)$$

where $A - 1 \leq \nu_{max} < A$. However, the eigenfunctions are now different and they are now given by

$$\psi_\nu(r) = N_\nu \frac{(\cosh r - 1)^{\frac{1}{2}(B-A)} (\cosh r + 1)^{-\frac{1}{2}(B+A)}}{2B \cosh r - 2A - 1} \hat{P}_{\nu+1}^{(\alpha, \beta)}(\cosh r) \quad (6)$$

where $\alpha = B - A - \frac{1}{2}$, $\beta = -B - A - \frac{1}{2}$. Here

$$N_\nu = -2^{A+2} B \left(\frac{\nu!(2A - 2\nu)(B + A - \nu + \frac{1}{2})\Gamma(B + A - \nu - \frac{1}{2})}{(B - A + \nu + \frac{1}{2})\Gamma(B - A + \nu - \frac{1}{2})\Gamma(2A - \nu + 1)} \right)^{\frac{1}{2}}, \quad (7)$$

is the normalization constant, and $\hat{P}_{\nu+1}^{(\alpha, \beta)}$ is $(\nu + 1)$ th-degree X_1 - Jacobi Polynomial.

The X_1 Jacobi polynomial is related to the usual Jacobi polynomial as [2]:

$$\hat{P}_\nu^{(\alpha, \beta)}(r) = -\frac{1}{2}(x - b)P_{\nu-1}^{(\alpha, \beta)}(r) + \frac{bP_{\nu-1}^{(\alpha, \beta)}(r) - P_{\nu-2}^{(\alpha, \beta)}(r)}{(\alpha + \beta + 2\nu - 2)} \quad (8)$$

where $b = \frac{\beta + \alpha}{\beta - \alpha}$

Using (8) the X_1 Jacobi Polynomial $\hat{P}_{\nu+1}^{(\alpha, \beta)}(\cosh r)$ can be written as

$$\hat{P}_{\nu+1}^{(\alpha, \beta)}(\cosh r) = \frac{1}{2(\alpha + \beta + 2\nu)} \left[\{(b - \cosh r)(\alpha + \beta + 2\nu) + 2b\} P_\nu^{(\alpha, \beta)}(\cosh r) - 2P_{\nu-1}^{(\alpha, \beta)}(\cosh r) \right] \quad (9)$$

Usual Jacobi polynomial $P_\nu^{(\alpha, \beta)}(\cosh r)$ further can be written in terms of Hypergeometric function as :

$$P_\nu^{(\alpha, \beta)}(\cosh r) = \frac{\Gamma(\nu + \alpha + 1)}{\nu! \Gamma(1 + \alpha)} F(\nu + \alpha + \beta + 1, -\nu, 1 + \alpha; \frac{1 - \cosh r}{2}). \quad (10)$$

To get the scattering states for this system two modifications of the bound state wavefunctions have to be made[7]: (i) The second solution of the Schrödinger equation must be retained - it has been discarded for bound state problems since it diverged asymptotically. (ii) Instead of the parameter ν labeling the number of nodes, one must use the wavenumber k so that we get the asymptotic behavior in terms of $e^{\pm ikr}$ as $r \rightarrow \infty$. After considering the second solution, Eq. (10) becomes

$$P_{\nu}^{(\alpha, \beta)}(\cosh r) = \frac{\Gamma(\nu + \alpha + 1)}{\nu! \Gamma(1 + \alpha)} \left[C_1 F(\nu + \alpha + \beta + 1, -\nu, 1 + \alpha; \frac{1 - \cosh r}{2}) \right. \\ \left. + C_2 (\frac{1 - \cosh r}{2})^{-(\nu + \alpha + \beta + 1)} F(\nu + \beta + 1, -\nu - \alpha, 1 - \alpha; \frac{1 - \cosh r}{2}) \right], \quad (11)$$

where C_1, C_2 are arbitrary constants. Considering the boundary condition, i.e as $r \rightarrow 0$, $(\frac{1 - \cosh r}{2}) \rightarrow 0$, $\psi_{\nu}(r)$ tending to finite, the allowed solution is

$$P_{\nu}^{(\alpha, \beta)}(\cosh r) = \frac{\Gamma(\nu + \alpha + 1)}{\nu! \Gamma(1 + \alpha)} C_1 F(\nu + \alpha + \beta + 1, -\nu, 1 + \alpha; \frac{1 - \cosh r}{2}). \quad (12)$$

Now replace ν by $A + ik$ and use $\alpha + \beta = -2A - 1$ We get

$$P_{(A+ik)}^{(\alpha, \beta)}(\cosh r) = C_1 \frac{\Gamma(B + ik + 1/2)}{(A + ik)! \Gamma(B - A + 1/2)} F(-A + ik, -A - ik, B - A + 1/2; \frac{1 - \cosh r}{2}). \quad (13)$$

Using Eq. (13) in (9) we get $P_{(\nu+1)}^{(\alpha, \beta)}(\cosh r) = P_{(A+ik+1)}^{(\alpha, \beta)}(\cosh r)$.

Thus the scattering state wavefunctions becomes

$$\psi_k(r) = N_k \frac{(\cosh r - 1)^{\frac{1}{2}(B-A)} (\cosh r + 1)^{-\frac{1}{2}(B+A)}}{2B \cosh r - 2A - 1} P_{A+ik+1}^{(\alpha, \beta)}(\cosh r) \quad (14)$$

Now using the properties of hypergeometric function [8] i.e

$$F(\alpha, \beta, \gamma; z) = (1 - z)^{-\alpha} \frac{\Gamma(\gamma) \Gamma(\beta - \alpha)}{\Gamma(\beta) \Gamma(\gamma - \alpha)} F(\alpha, \gamma - \beta, \alpha - \beta + 1; \frac{1}{1 - z}) \\ + (1 - z)^{-\beta} \frac{\Gamma(\gamma) \Gamma(\alpha - \beta)}{\Gamma(\alpha) \Gamma(\gamma - \beta)} F(\beta, \gamma - \alpha, \beta - \alpha + 1; \frac{1}{1 - z}) \quad (15)$$

and taking the limit $r \rightarrow \infty$ the fourth term of the hypergeometric equation vanishes.

Finally we get the asymptotic form of (14), which is given as

$$\lim_{r \rightarrow \infty} \psi_k(r) = N_k \frac{C_1 2^{-2ik-3A} [(2ik - 1)aP + Qc]}{4B(2ik - 1)} \left[\frac{bP(1 - 2ik)2^{-4ik}}{aP(2ik - 1) + Qc} e^{ikr} - e^{-ikr} \right] \quad (16)$$

where

$$P = \frac{\Gamma(B + ik + 1/2)}{(A + ik)!\Gamma(B - A + 1/2)}; \quad Q = \frac{\Gamma(B + ik - 1/2)}{(A + ik - 1)!\Gamma(B - A + 1/2)};$$

$$a = \frac{\Gamma(B - A + 1/2)\Gamma(-2ik)}{\Gamma(-A - ik)\Gamma(B - ik + 1/2)}; \quad b = \frac{\Gamma(B - A + 1/2)\Gamma(2ik)}{\Gamma(-A + ik)\Gamma(B + ik + 1/2)};$$

$$c = \frac{\Gamma(B - A + 1/2)\Gamma(-2ik + 2)}{\Gamma(-A - ik + 1)\Gamma(B - ik + 3/2)}; \quad d = \frac{\Gamma(B - A + 1/2)\Gamma(2ik - 2)}{\Gamma(-A + ik - 1)\Gamma(B + ik - 1/2)};$$

The asymptotic behavior for the radial wavefunction (for $l=0$) is given by [1]

$$\lim_{r \rightarrow \infty} \psi_k(r) \simeq \frac{1}{2k} [S_{l=0} e^{ikr} - e^{-ikr}] \quad (17)$$

From (16) and (17) we get

$$S_{l=0} = \frac{bP(1 - 2ik)2^{-4ik}}{aP(2ik - 1) + Qc} \quad (18)$$

Using P, Q, a, b and c, we get after simple calculation (using $\Gamma(n + 1) = n\Gamma(n)$)

$$S_{l=0} = S_{l=0}^{GPT} \frac{[B^2 - (ik - 1/2)^2]}{[B^2 - (ik + 1/2)^2]} \\ = \frac{\Gamma(2ik)\Gamma(-A - ik)\Gamma(B - ik + 1/2)2^{-4ik}}{\Gamma(-A + ik)\Gamma(-2ik)\Gamma(B + ik + 1/2)} \frac{[B^2 - (ik - 1/2)^2]}{[B^2 - (ik + 1/2)^2]}. \quad (19)$$

Thus we notice that the scattering amplitudes for the two potentials (i.e. GPT and new X_1 SIP) are different even though the bound state spectrum is identical for them. It will be interesting to see how the scattering amplitude change as we go from the potential here to the one whose eigenfunctions are in terms of X_n Jacobi polynomials.

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